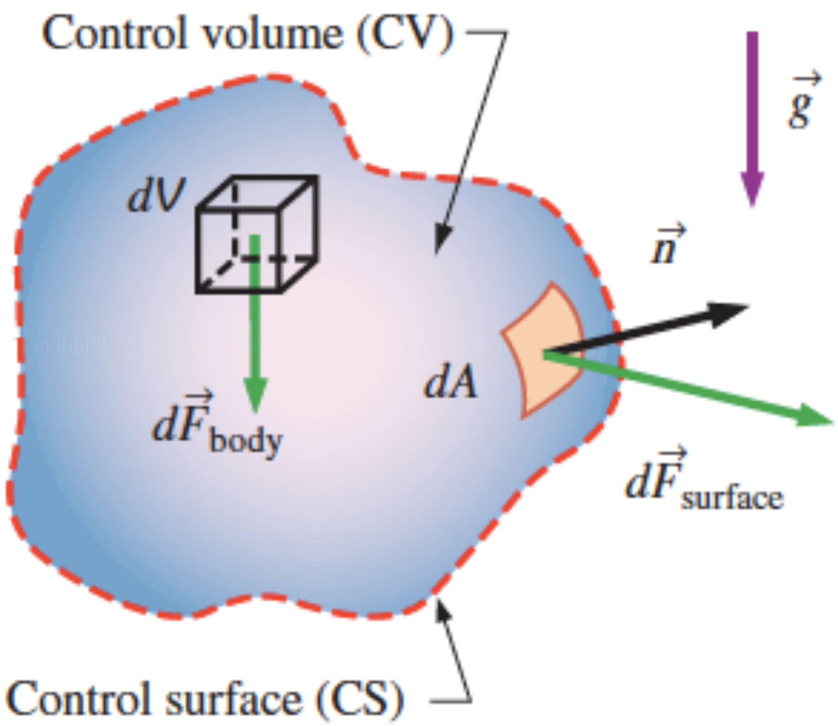
Euler Equation and Bernoulli's Theorem

Overview

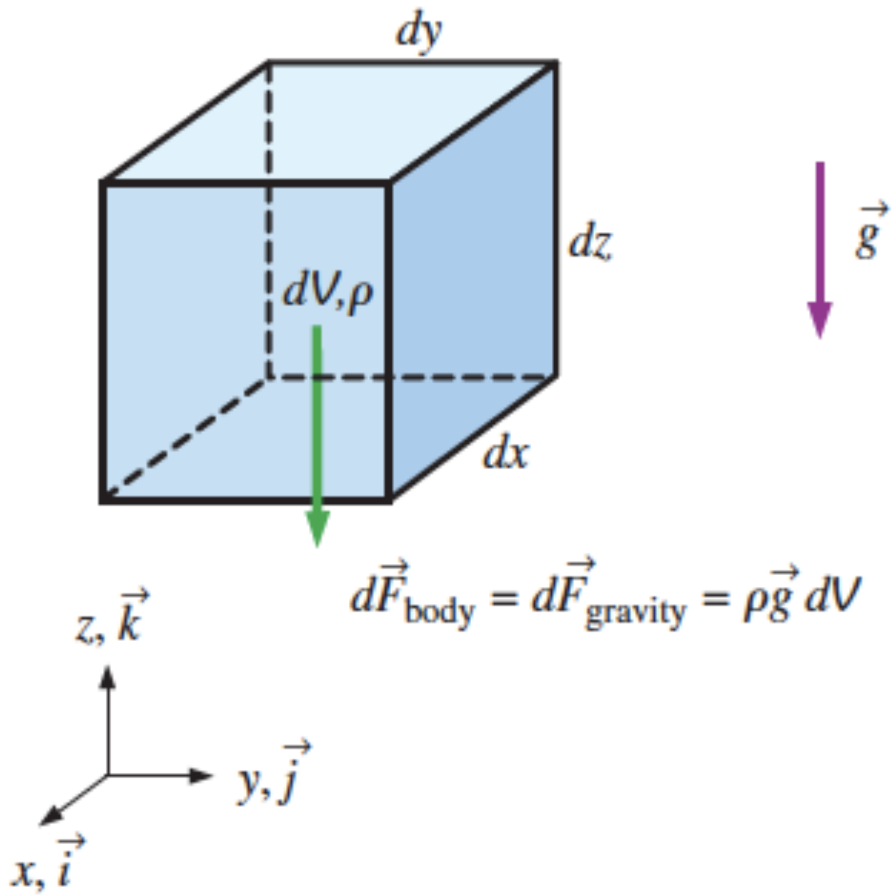
- Fluid dynamics deals with the relation between the motion of fluids considering the forces and moments which create the motion.
- Forces acting on fluid elements are of two types: body forces acting on the center of mass of the fluid element and surface forces acting through the surfaces.
- For ideal (bulk) fluids (zero viscosity and compressibility) the surface forces reduce to an isotropic pressure (Pascal's Theorem) and the governing dynamical equation was derived by Euler.
- Euler's equation is a PDE that can be integrated along the streamlines and the integral is known as Bernoulli's equation (Bernoulli's first Theorem).

Forces on a control volume

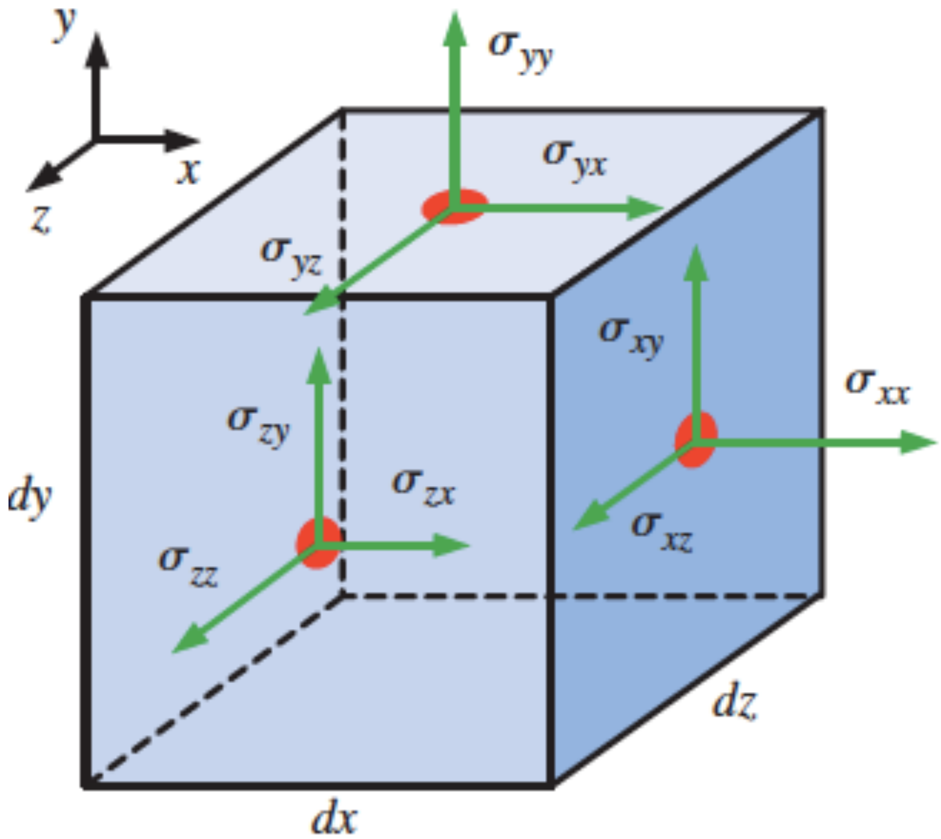


The forces acting on a control volume consist of body forces that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and surface forces that act on the control surface (such as pressure and viscous forces and reaction forces at points of contact).

Body forces: 1 vector or rank 1 tensor



Surface forces: 2 vectors or rank 2 tensor



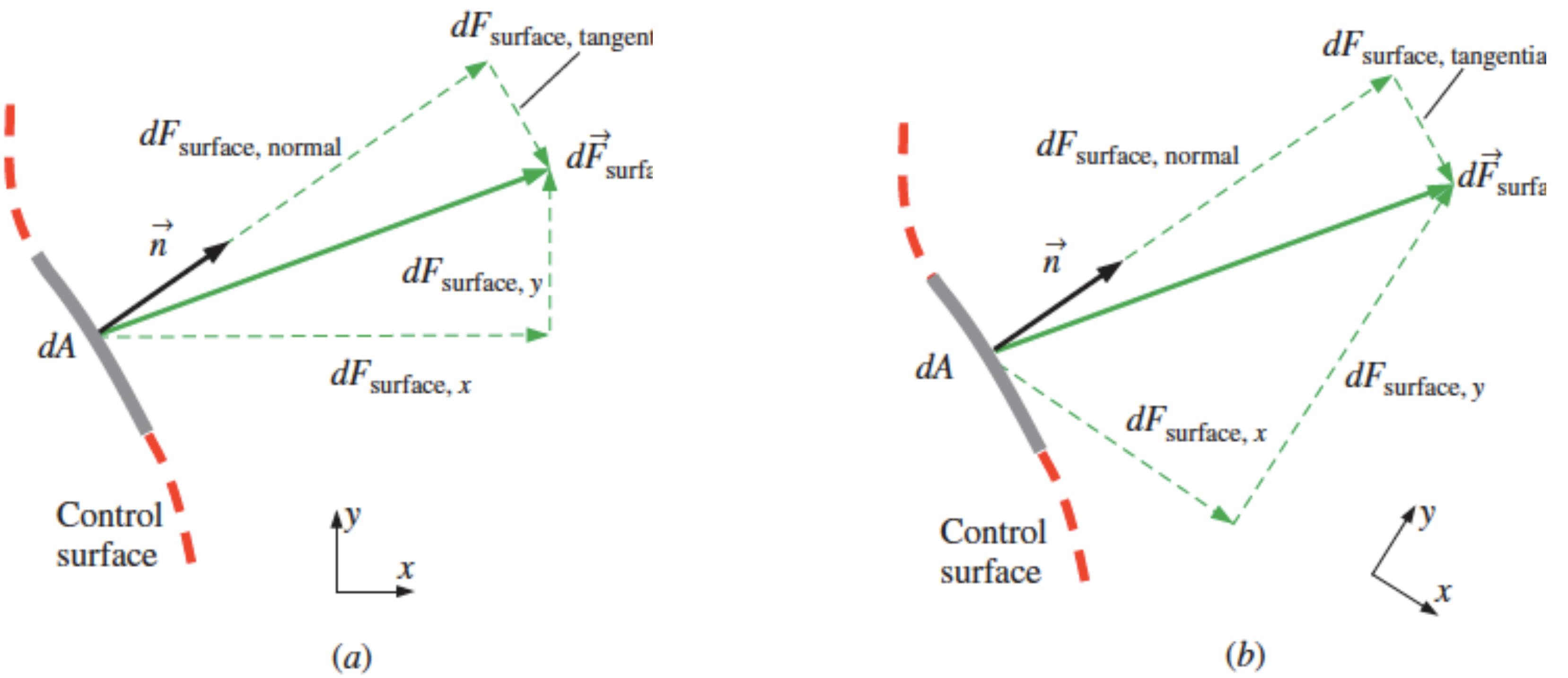
σ_{ij} is the force per unit area in the direction of j through the plane (perpendicular to) i

Stress tensor in cartesian coordinates

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

The diagonal components of the stress tensor are called normal stresses; they are composed of pressure (which always acts inwardly normal) and viscous stresses.
Viscous stresses are discussed in more detail later. The off-diagonal components, are called shear stresses; since pressure can act only normal to a surface, shear stresses are composed entirely of viscous stresses.

Surface forces



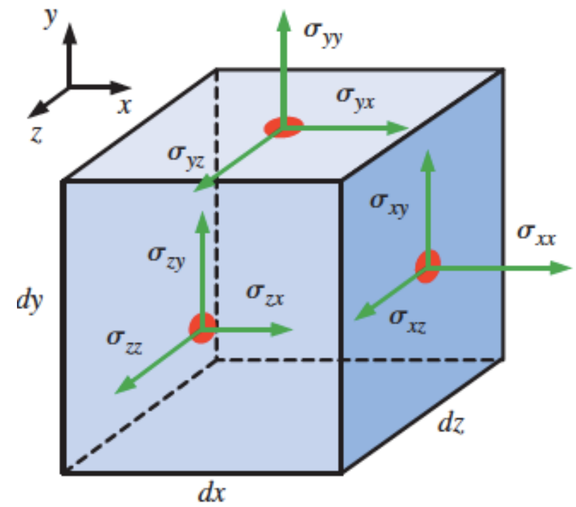
Surface force acting on a differential surface element: $d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n}\, dA$

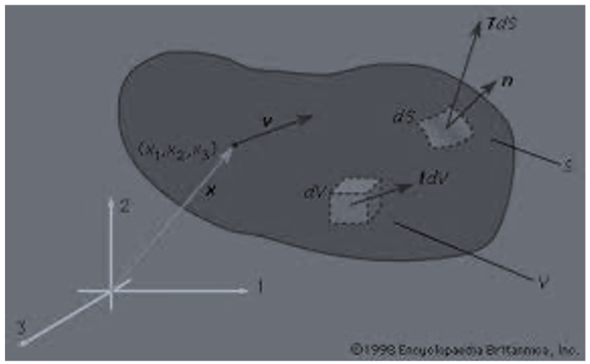
Stress on surface element dA, with $n \equiv (n_x, n_y, n_z)$

From $\sigma_{ij} \cdot \vec{n}$ we find the stress components:

- in the x direction, $(\sigma_{xx} + \sigma_{yx} + \sigma_{zx})n_x$
- in the y direction, $(\sigma_{xy} + \sigma_{yy} + \sigma_{zy})n_y$
- in the z direction, $(\sigma_{xz} + \sigma_{yz} + \sigma_{zz})n_z$

The force is the product of the stress by the area.





Total force

Body forces act on each volumetric portion of the control volume. The body force acts on a differential element of fluid of volume dV within the control volume, and we must perform a volume integral to account for the net body force on the entire control volume.

Total body force acting on control volume: $$\sum \vec{F}_{\text{body}} = \int_{CV} \rho \vec{g}\, dV = m_{CV}\vec{g}$$

Surface forces act on each portion of the control surface. A differential surface element of area dA and unit outward normal $\vec{n}$ is shown, along with the surface force acting on it. We must perform an area integral to obtain the net surface force acting on the entire control surface.

Total surface force acting on control surface: $$\sum \vec{F}_{\text{surface}} = \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA$$

The linear momentum equation

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{sys}} \rho \vec{V}\, dV$$

- Therefore, Newton's second law can be stated as the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.

- Applying the Reynolds transport theorem we find:

$$\frac{d(\ \vec{P}\ )_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho \vec{V}\, dV + \int_{\text{CS}} \rho \vec{V}\,(\vec{V}_r \cdot \vec{n})\, dA$$

Newton's law for a control volume

General: $$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

which is stated in words as

$$\begin{pmatrix}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the linear momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

Fixed CV: $$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V} \cdot \vec{n})\, dA$$

Water jet on stationary plate

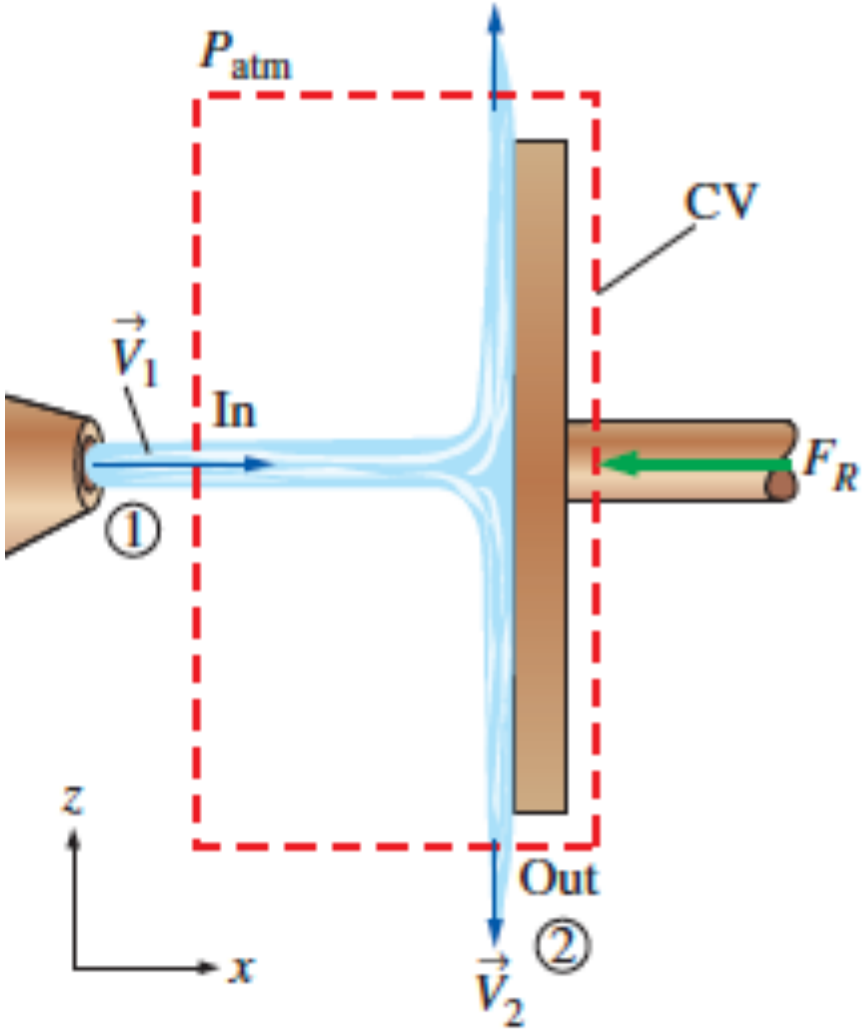
- The momentum equation for steady flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

- The reaction force at the plate is

$$-F_R = 0 - \beta \dot{m} V_1$$

Note that $\beta = 1$in this course.



Deceleration of a spacecraft

The spacecraft is treated as a body with constant mass, and the momentum equation is

$$\vec{F}_{\text{thrust}} = m_{\text{spacecraft}}\,\vec{a}_{\text{spacecraft}} = \sum_{\text{in}} \beta \dot{m} \vec{V} - \sum_{\text{out}} \beta \dot{m} \vec{V}$$

Since the motion is 1d, the acceleration is

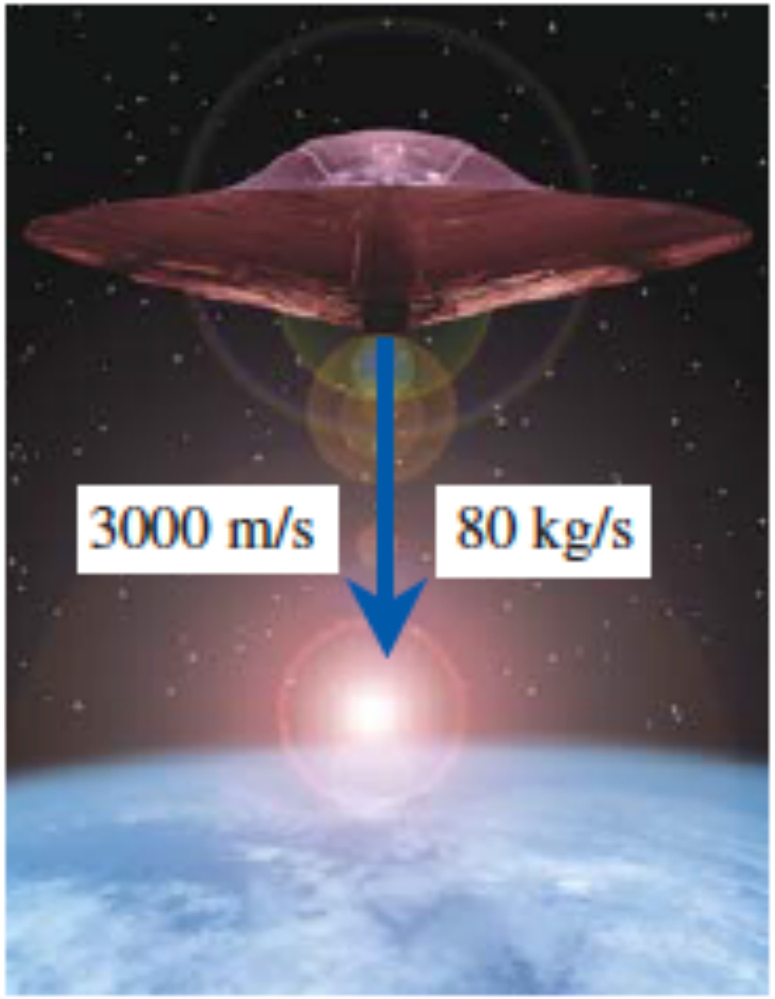
$$a_{\text{spacecraft}} = \frac{dV_{\text{spacecraft}}}{dt} = -\frac{\dot{m}_{\text{gas}}}{m_{\text{spacecraft}}} V_{\text{gas}}$$

The velocity change of the spacecraft is

$$dV_{\text{spacecraft}} = a_{\text{spacecraft}}\,dt \rightarrow \Delta V_{\text{spacecraft}} = a_{\text{spacecraft}}\,\Delta t$$

The thrusting force exerted on the space aircraft is,

$$F_{\text{thrust}} = 0 - \dot{m}_{\text{gas}} V_{\text{gas}}$$

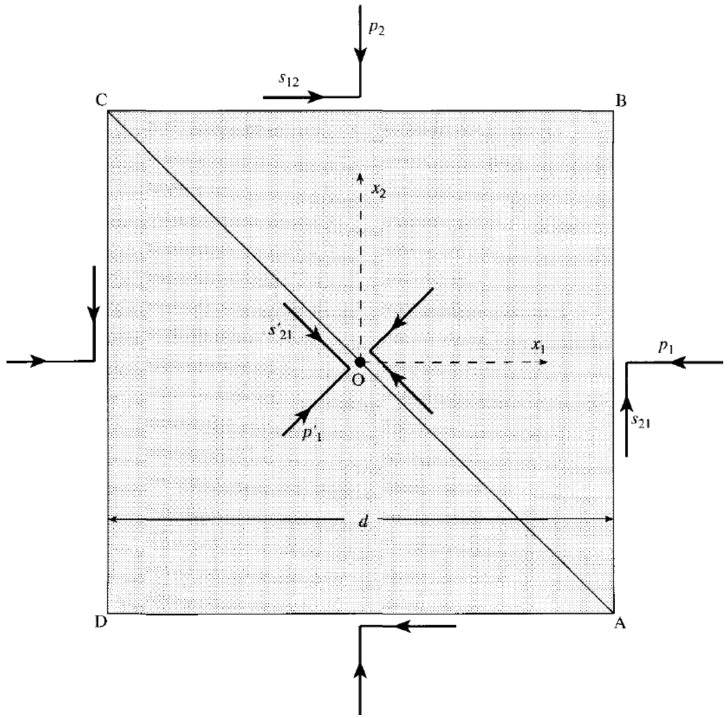


Note that $\beta = 1$in this course.

More on the stress tensor
Pascal’s Theorem

The stress tensor is symmetric (Faber or Batchelor)

- Choose axes paralel to the directions of a cubic fluid element (dashed lines)
- Consider a cubic fluid element of side *d*
- Consider one face of the cube, *ABCD.* The plane that passes through *AB* has a normal in the *x (1)* direction.



Balance of forces and torques

- Forces

Normal forces are: $d^2\, p_1$, $d^2\, p_2$ and $d^2\, p_3$ with three other forces acting on the opposite faces.

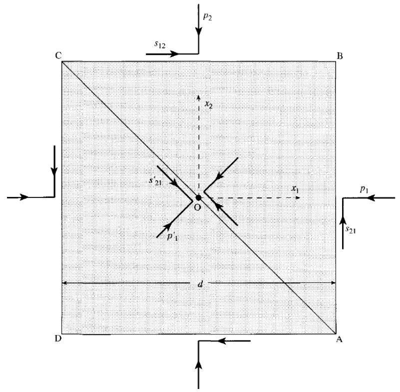
The forces act on an element of fluid of volume d^3. As $d \to 0$ the acceleration diverges as $\frac{1}{d}$ unless the forces on opposite faces of the cube balance.

- Torques

Tangencial forces: $d^2\, s_{21}$ and $d^2\, s_{12}$ produce a torque $d^3\, (s_{21} - s_{12})$ about O, which produces an angular acceleration that diverges as $\frac{1}{d^2}$ (the moment of inertia of the element of fluid scales with the 5th power of *d*) unless the torque vanishes. So, $s_{21} = s_{12} = s_3$ (Faber's notation, 3 is the axis of rotation).

Transformation of the stress tensor under rotation of the axes

- Rotate the axes by 45º, along the diagonal of the square face and consider the plane that passes through *AC.* Note that the length of *AC* is $\sqrt{2}d$. The height of the triangular prism *ABC* is d.
- Consider the normal and tangencial forces in the new primed (‘) directions. Since both the sine and the cosine of 45º are $\frac{\sqrt{2}}{2}$, simple trigonometry yields,



$$2p_1' = p_1 + p_2 - 2s_3,$$
$$2s_3' = p_1 - p_2.$$

- Similarly,

$$2p_2' = p_1 + p_2 + 2s_3,$$

and thus the sum of the two normal stresses is invariant,

$$p_1' + p_2' = p_1 + p_2.$$

Average pressure

- A similar calculation involving p_3 leads to the important result, that the average pressure is an invariant, i.e. the same for any rotation.

$$p = \frac{1}{3}(p_1 + p_2 + p_3)$$

- Note that the p_i's may differ and will change as the axes rotate, but their sum does NOT.

Fluids at mechanical equilibrium

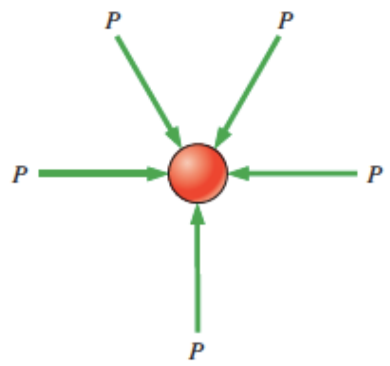
- At equilibrium the shear stresses vanish (otherwise there would be flow) which implies that,

$$p_1' = p_1 = p_2,$$

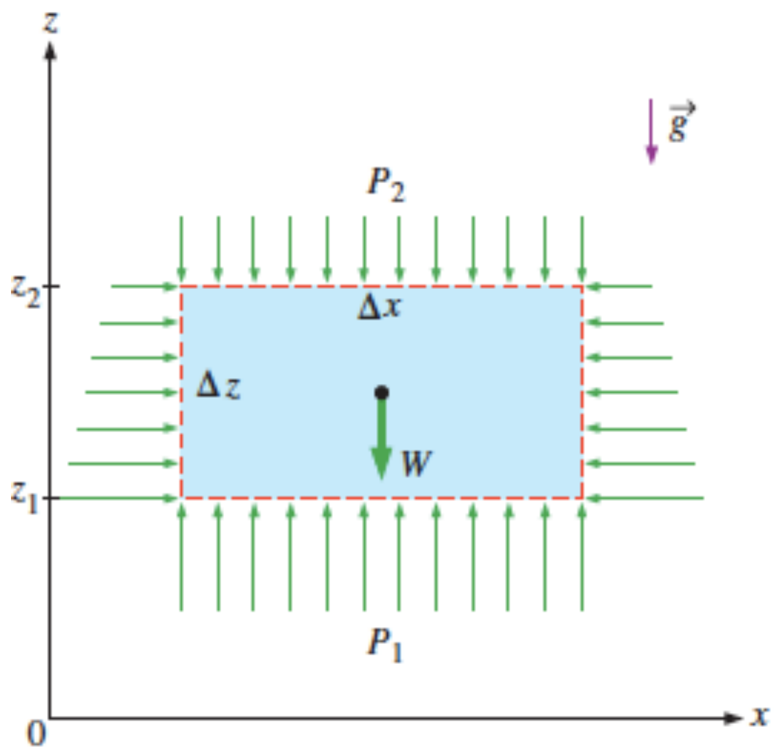
and similarly, for p_3. This means that $p_1 = p_2 = p_3 = p$ in any frame of reference, i.e. the pressure is a scalar field.

Pascal's Theorem

At mechanical equilibrium the pressure is a scalar field, $p(\vec{r})$.



Increase of pressure with depth

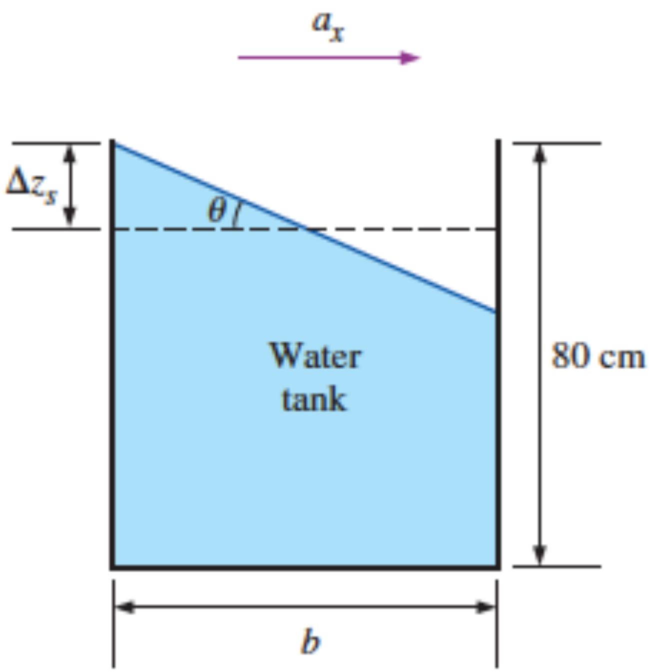


$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g \, dz$$

$$P = P_{atm} + \rho g h \quad \text{or} \quad P_{gage} = \rho g h$$

Rigid body motion I (overflow from a water tank)

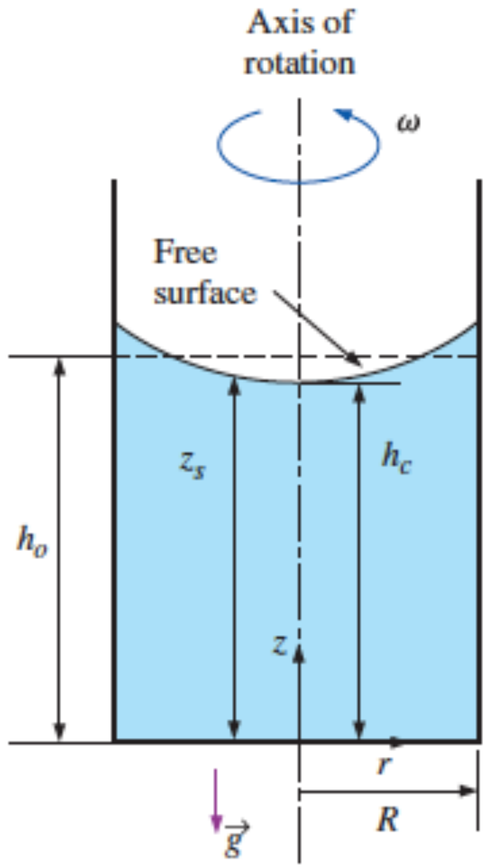
Rigid-body motion of fluids: $\vec{\nabla} P + \rho g \vec{k} = -\rho \vec{a}$



Pressure variation: $P = P_0 - \rho a_x x - \rho(g + a_z)z$

Surfaces of constant pressure: $\frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = \text{constant}$

Rigid body motion II (free surface of a rotating vertical cylinder)



In cylindrical coordinates,

$$\frac{\partial P}{\partial r} = \rho r \omega^2, \quad \frac{\partial P}{\partial \theta} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho g$$

and

$$dP = \rho r \omega^2\, dr - \rho g\, dz$$

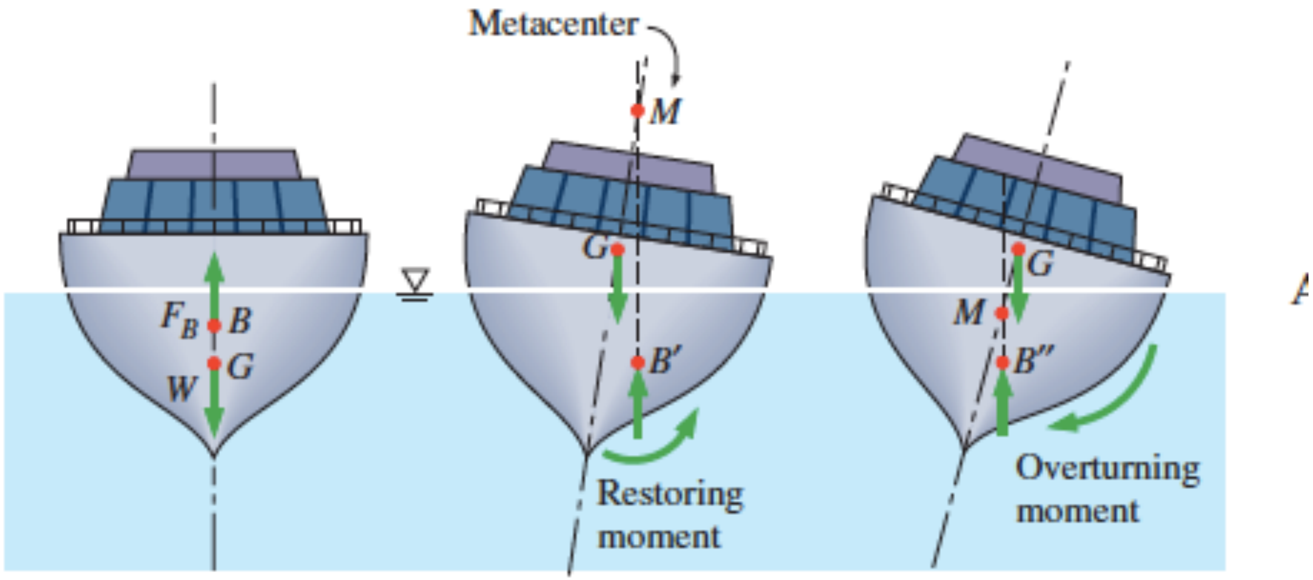
From *dP=0*, we find the isobars

Surfaces of constant pressure: $$z_{\text{isobar}} = \frac{\omega^2}{2g} r^2 + C_1$$

Mass conservation yields, $h_c = h_0 - \dfrac{\omega^2 R^2}{4g}$

Free surface: $$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

Stability of floating objects



(*a*) Stable (*b*) Stable (*c*) Unstable

A floating body is stable if the body is (a) bottom-heavy and thus the center of gravity G is below the centroid B of the body, or (b) if the metacenter M is above point G. However, the body is (c) unstable if M is below point G.

The Euler fluid: zero viscosity and zero compressibility

As a result the shear stresses are zero and the density is constant

Euler fluid

- For an Euler fluid the continuity equation implies that

$$\nabla \cdot \vec{V} = 0$$

and the inviscid (zero viscosity) condition implies that the stress tensor reduces to a scalar isotropic pressure, *p,* which may vary in space (pressure field).

- The surface forces acting on an element of fluid, per unit volume, are given by $-\nabla p$ (recall that the force in the x direction is: $-\frac{dp(x)dydz}{dV} = -\frac{\partial p}{\partial x}$).
- The surface forces per unit mass are then $-\frac{\nabla p}{\rho}$.

- The total force may include body terms, such as gravity, $-\nabla gz$.

- The Euler equation is

$$\vec{f} = -\frac{\nabla p}{\rho} - \nabla gz = \frac{D\vec{V}}{Dt}$$

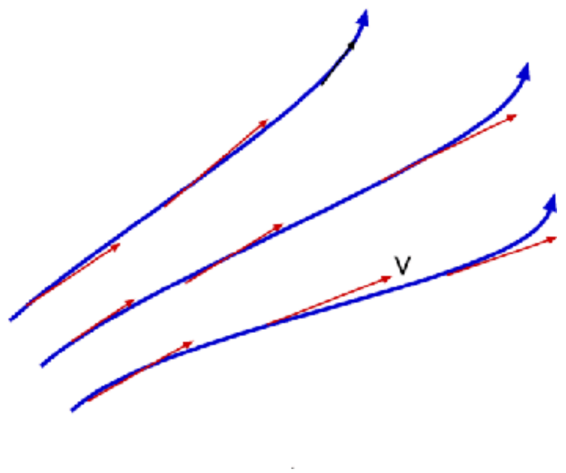
Stationary flow

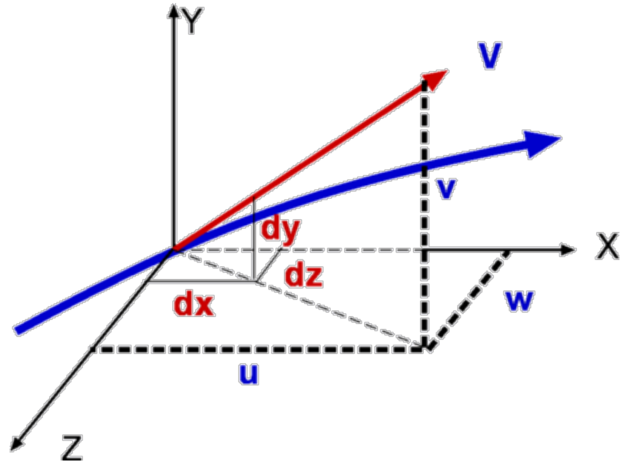
- Recall that the material derivative is given by (Euler or velocity field description of the flow)

$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \vec{\nabla}\right)\vec{V}$$

- And note that the acceleration is generally non-zero for a stationary flow.

Longitudinal and transverse componentes of the acceleration





- Consider the longitudinal component (1) of the velocity field along a streamline. Only that componente (1) of ***u*** is non-zero. The longitudinal acceleration, in the direction of the flow, is

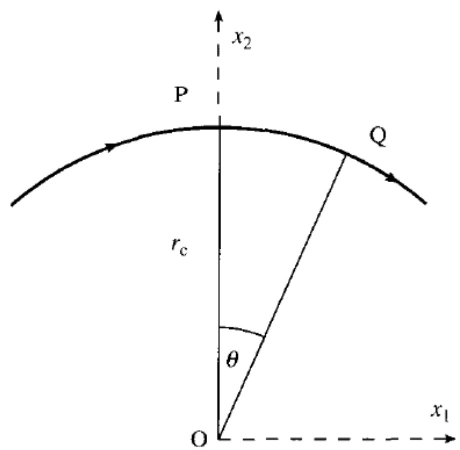
$$(\boldsymbol{u}\cdot\nabla)u_1 = u_1\frac{\partial u_1}{\partial x_1} = \frac{\partial(\frac{1}{2}u^2)}{\partial l}$$

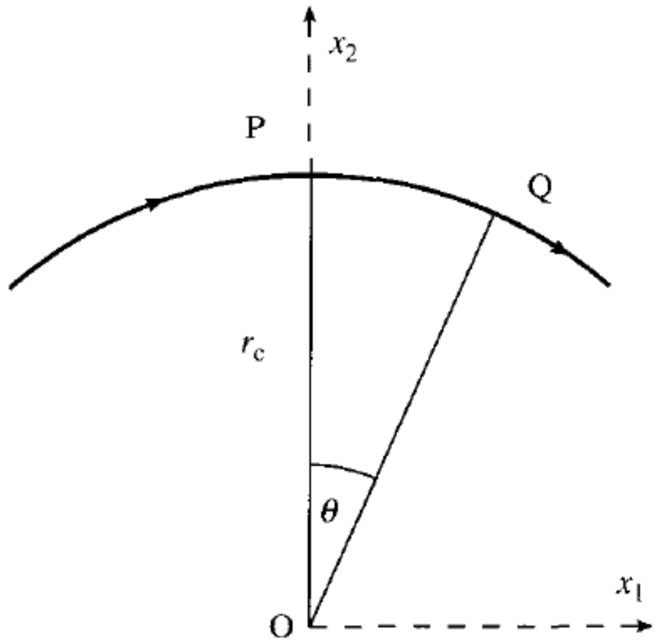
Longitudinal and transverse componentes pf the acceleration

- The acceleration will have in general transverse components (e.g. centripetal acceleration) in directions 2 and 3, namely,

$$u_1(\partial u_2/\partial x_1) \text{ and } u_1(\partial u_3/\partial x_1) \text{ in the } x_2 \text{ and } x_3 \text{ directions}$$

- Assume that second term is zero. Then the motion around point P is in the plane 12 (no motion in the third direction).





Let O be the centre of curvature at P and r_c its radius.

At nearby Q the second component of the velocity is,

$$u_{2,Q} \approx \left(\frac{\partial u_2}{\partial x_1}\right)_P r_c \sin\theta + \left(\frac{\partial u_2}{\partial x_2}\right)_P r_c(1 - \cos\theta),$$

Which is to first order in θ,

$$u_{2,Q} \approx \left(\frac{\partial u_2}{\partial x_1}\right)_P r_c\theta$$

Note that to first order in θ:

$$u_{2,\mathrm{Q}} = u_{\mathrm{Q}} \sin\theta \approx u_{\mathrm{P}}\theta,$$

And then, we find

$$\left(\frac{\partial u_2}{\partial x_1}\right)_{\mathrm{P}} = \left(\frac{u}{r_{\mathrm{c}}}\right)_{\mathrm{P}},$$

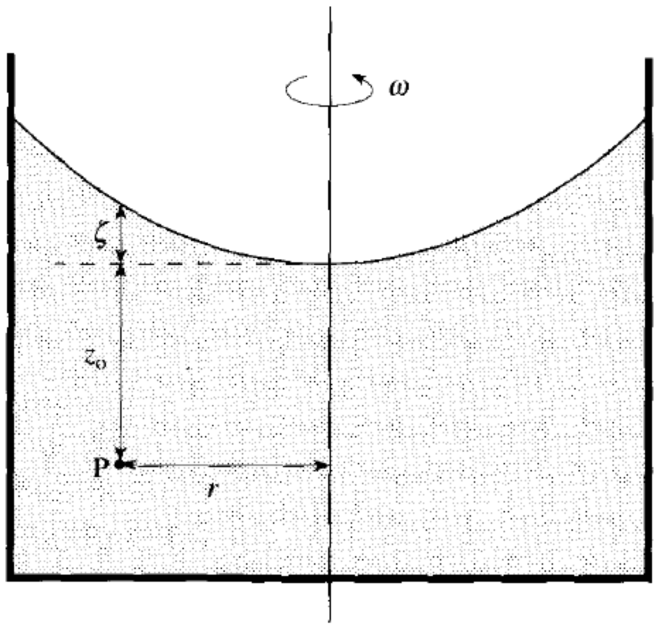
Which is equivalent to the familiar centripetal acceleration result (used before):

$$u_1 \left(\frac{\partial u_2}{\partial x_1}\right)_{\mathrm{P}} = \left(\frac{u^2}{r_{\mathrm{c}}}\right)_{\mathrm{P}}.$$

The Euler equation (transverse pressure gradient) becomes:

$$\frac{\partial p}{\partial r} = \frac{\rho u^2}{r_{\mathrm{c}}}.$$

The bucket of water revisited (Faber)



The hydrostatic pressure at P is

$$p = p_A + \rho g(z_o + \zeta),$$

Using the expression for the transverse pressure gradient, the rate of change of ζ at *r*, becomes

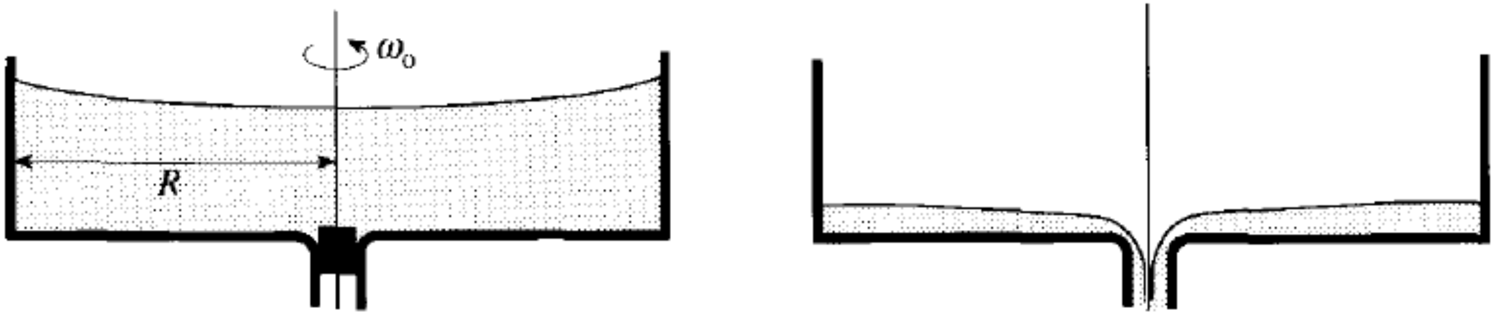
$$\frac{d\zeta}{dr} = \frac{u^2}{rg} = \frac{\omega^2 r}{g},$$

Integration, yields for the height at *r*:

$$\zeta = \frac{\omega^2 r^2}{2g}.$$

The plug-hole vortex (Faber)

Conservation of angular momentum:
particles initially at *R* move to *r,* with angular velocity $\omega \approx \omega_o \frac{R^2}{r^2}.$



(a) (b)

Using the expression for the transverse pressure gradient, $$\frac{\partial p}{\partial r} \approx \rho\omega^2 r \approx \frac{\rho\omega_o^2 R^4}{r^3},$$

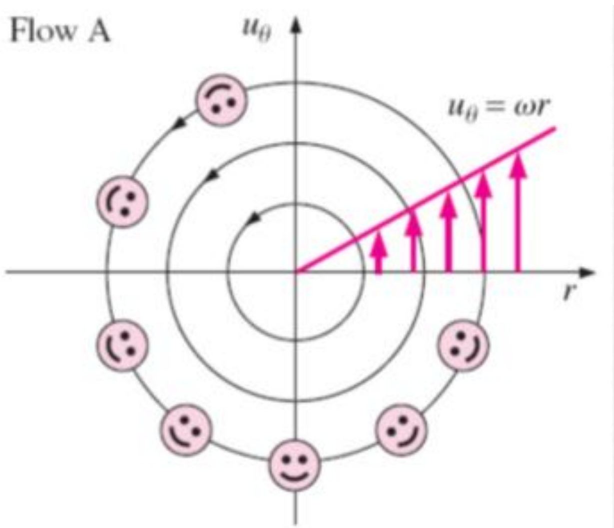
Integration, yields for the depth at *r* (measured from the height at R): $$\zeta \approx \frac{\omega_o^2 R^4}{2r^2 g}.$$

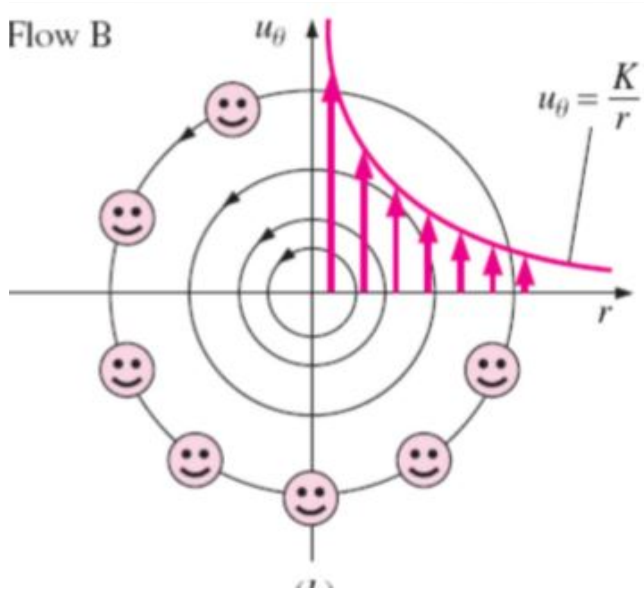
Vorticity $\Omega = \nabla\times\vec{V}$

Using cylindrical coordinates, or otherwise, it is found that the vorticity in the previous two examples is given by (see Faber)

$$\Omega = \omega + \frac{\mathrm{d}(\omega r)}{\mathrm{d}r} = 2\omega + r\frac{\mathrm{d}\omega}{\mathrm{d}r},$$

The rigid body flow in the bucket (A) is rotational, while that in the plug vortex (B) is irrotational.





Both the efflux of the water and the trajectory of the resulting jet are well described by ideal fluid theory

Another spectacular success is the theory of flight. The ideal flow of air around a wing is able to describe the lift necessary for flight, and much more.

Bernoulli equation

The longitudinal (along a streamline) component of the acceleration $\partial(\frac{1}{2}u^2)/\partial l$,

and that of the force $-\dfrac{1}{\rho}\dfrac{\partial p}{\partial l} - \dfrac{\partial(gz)}{\partial l}.$

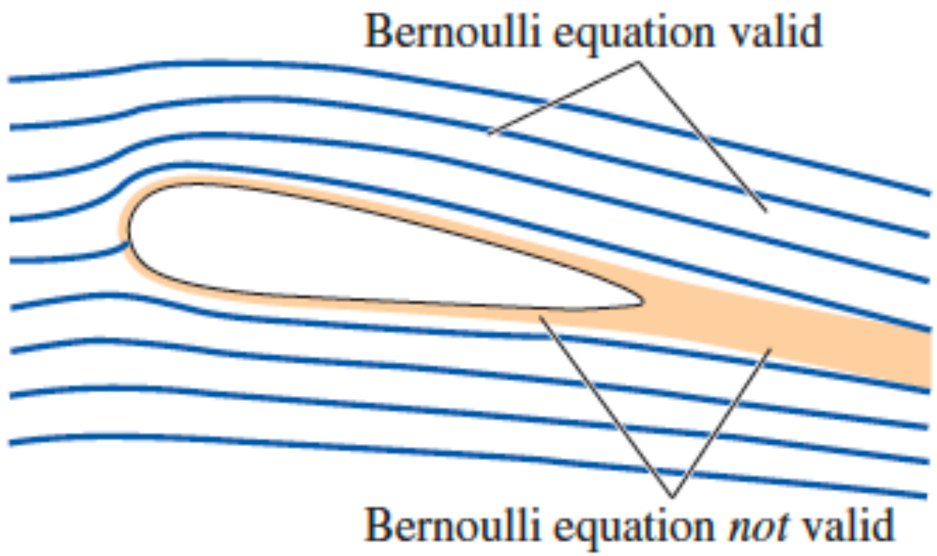
may be integrated along the streamline, to give

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2 = \text{constant}.$$

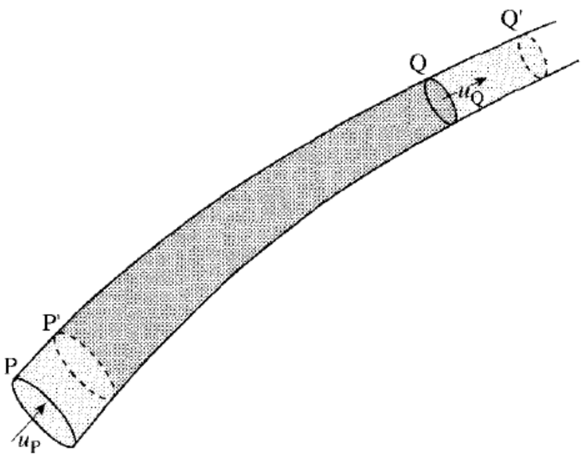
which is known as the Bernoulli equation. Note that the constant of integration may (and will in general) depend on the streamline.

The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces.

Such regions occur outside of boundary layers and wakes.



Bernoulli equation (conservation of energy)



The variation of kinetic and potential energy between points P and Q is

$$\Delta m \left\{ \left(\frac{1}{2} u^2 \right)_Q - \left(\frac{1}{2} u^2 \right)_P + (gz)_Q - (gz)_P \right\}.$$

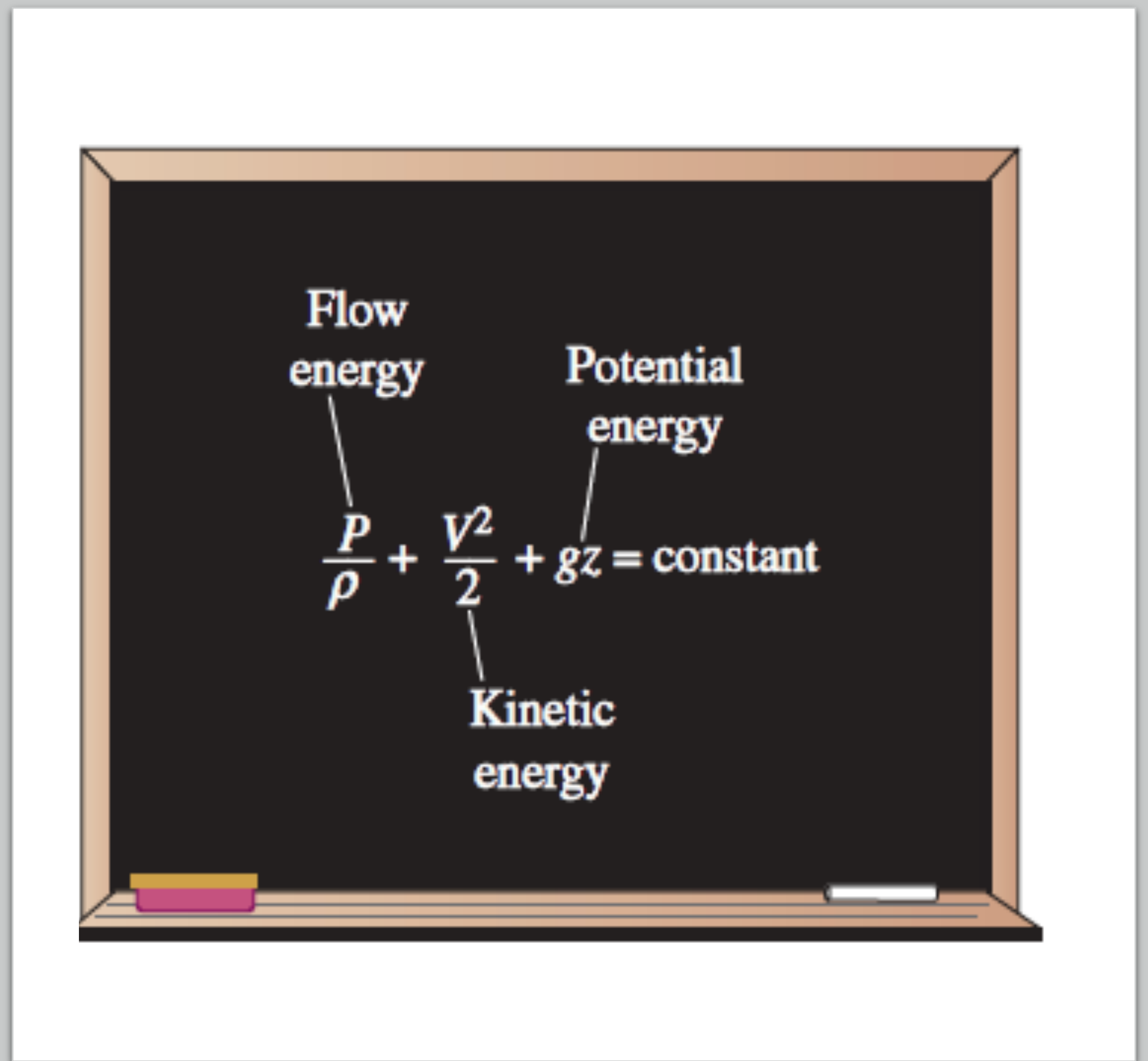
The work done on the fluid is

$$\frac{\Delta m}{\rho} (p_P - p_Q).$$

Equating the two we find (again) the Bernoulli equation.

$$\frac{p}{\rho} + gz + \frac{1}{2} u^2 = \text{constant}.$$

The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow when compressibility and frictional effects are negligible.



Bernoulli equation for flows with zero vorticity: constant independente of the streamline

Zero vorticity implies:

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2}, \quad \frac{\partial u_3}{\partial x_1} = \frac{\partial u_1}{\partial x_3},$$

Then, the longitudinal component of the material derivative is

$$\begin{aligned}(\boldsymbol{u} \cdot \boldsymbol{\nabla}) u_1 &= u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} \\ &= u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_1} + u_3 \frac{\partial u_3}{\partial x_1} \\ &= \frac{\partial \left(\frac{1}{2} u^2\right)}{\partial x_1}.\end{aligned}$$

With similar results for the transverse directions.

Then, $$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\left(\frac{1}{2}\, u^2\right).$$

And the Euler equation becomes:

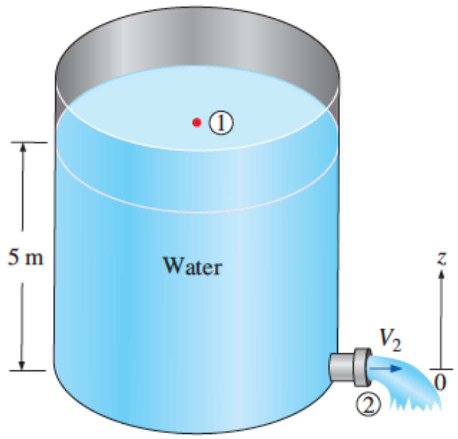
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{p}{\rho} + gz + \frac{1}{2}\, u^2\right) = 0$$

In stationary flow,

$$\boldsymbol{\nabla}\left(\frac{p}{\rho} + gz + \frac{1}{2}\, u^2\right) = 0,$$

The Euler equation may be integrated in ANY direction, implying that the integration constant is the SAME for all the streamlines (that is everywhere).

Velocity of discharge from a large tank



Direct application of Bernoulli's equation,

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \left(V_1^2 \approx 0,\ z_2 = 0\right) \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

The discharge velocity is,

$$V_2 = \sqrt{2gz_1}$$

Table top experiment

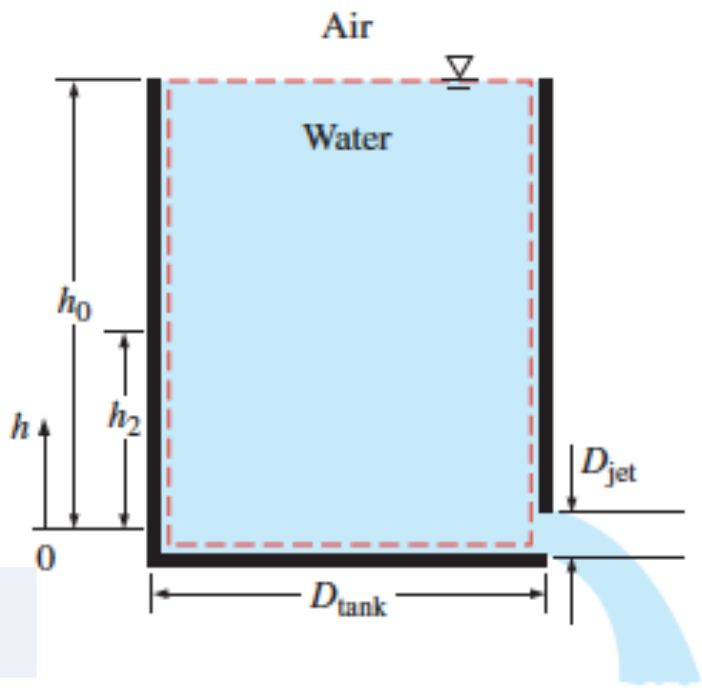
Try to check this for water and oil.

Time of discharge of a tank

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

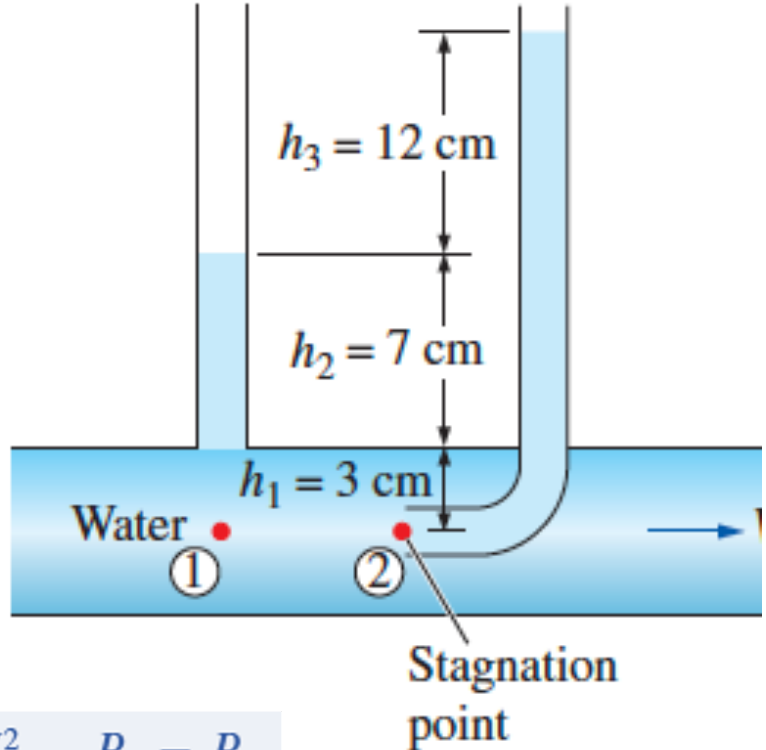
$$\dot{m}_{out} = (\rho V A)_{out} = \rho\sqrt{2gh}A_{jet}$$

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank}h)}{dt} \to -\rho\sqrt{2gh}(\pi D_{jet}^2/4) = \frac{\rho(\pi D_{tank}^2/4)dh}{dt}$$



$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \to t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

Pitot tube



Hydrostatic pressure at points 1 and 2,

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

Application of Bernoulli's equation,

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\rho g} + \cancel{\frac{V_2^2}{2g}}^{\,0} + \cancel{z_2} \rightarrow \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

Gives for the for the velocity at 1,

$$V_1 = \sqrt{2gh_3}$$